Forbidden Positions and Rook Polynomials

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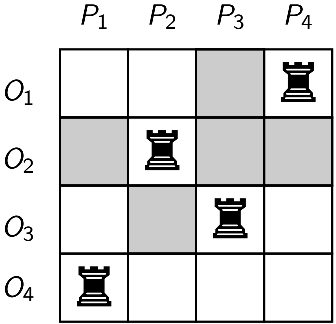
MATH 4620H

Problem

How many ways can we arrange a set of n objects, if there are restrictions on the positions each object can be in?

Problem Setup

We represent this problem as a board, with darkened squares representing the invalid positions for each object. We are interested in ways we can position non-capturing rooks on the board, where we can only place them on the light squares.



This represents the permutation $O_4 O_2 O_3 O_1$

Rook Polynomial

Let $r_k(B)$ be the number of ways to place k non-capturing rooks on the *darkened squares* of a board B. The **rook polynomial** $R(x, B)$ is the generating function of $r_k(B)$:

$$R(x, B) = \sum_k r_k(B)x^k.$$

Note that for any any board B, $r_0(B) = 1$.

Theorem

Let $r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_k(B)x^k$ be the rook polynomial for the darkened squares of an $n \times n$ board B. Then, the number of ways to place n rooks on the light squares of B is counted by

$$n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)!.$$

Proof

Use the Principle of Inclusion-Exclusion and let P_i be the property that there are at least i rooks in forbidden positions.

There are $r_i(B)$ ways to place i rooks in restricted positions. Then, there are $(n-i)!$ to arrange the rest of the rooks without consideration for whether their positions are restricted or not.

Therefore for all $i \in \mathbb{N}$, $P_i = r_i(B)(n-i)!$, allowing us to arrive at

$$n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)!.$$

Theorem

Let $R(x, B_1)$, $R(x, B_2)$ be the rook polynomials of disjoint subboards B_1, B_2 of a board B. Then

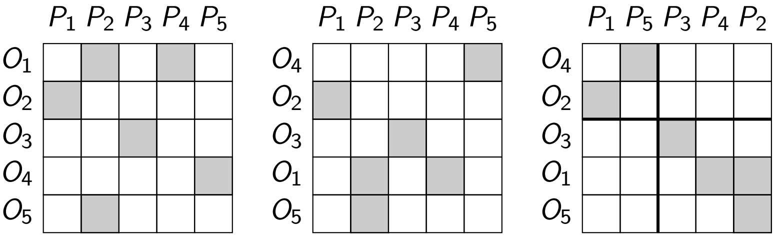
$$R(x, B) = R(x, B_1)R(x, B_2).$$

This idea can be extended to any number of disjoint subboards, thus

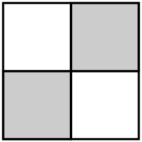
$$R(x, B) = R(x, B_1)R(x, B_2) \cdots R(x, B_k)$$

for disjoint subboards $B_1, \ldots, B_k$.

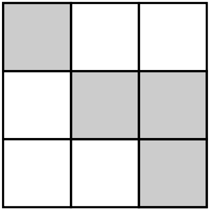
Example


P_1 P_2 P_3 P_4 P_5
O_1 O_2 O_3 O_4 O_5
P_1 P_2 P_3 P_4 P_5
O_4 O_2 O_3 O_1 O_5
P_1 P_5 P_3 P_4 P_2
O_4 O_2 O_3 O_1 O_5

Subboard Rook Polynomials



$r_0(B_1) = 1,\ r_1(B_1) = 2,\ r_2(B_2) = 1$
$R(x, B_1) = 1 + 2x + x^2$



$r_0(B_2) = 1,\ r_1(B_2) = 4,\ r_2(B_2) = 4,$
$r_3(B_2) = 1$
$R(x, B_2) = 1 + 4x + 4x^2 + x^3$

Rook Polynomial of B

Then by the disjoint subboard theorem, we get that the rook polynomial for the original board B is

$$\begin{aligned} R(x,B) &= R(x,B_1)R(x,B_2) = (1+2x+x^2)(1+4x+4x^2+x^3) \\ &= 1+6x+13x^2+13x^3+6x^4+x^5 \end{aligned}$$

and we get

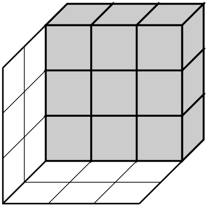
$$5! - 6\times 4! + 13\times 3! - 13\times 2! + 6\times 1! - 1\times 0! = 33$$

ways of placing 5 rooks on the light squares of B so that they are non-capturing.

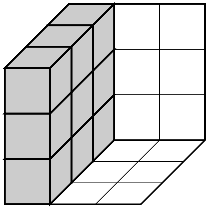
Rook Polynomials in Three Dimensions

To be non-capturing, each rook must be the only rook in its slab, wall, and layer.

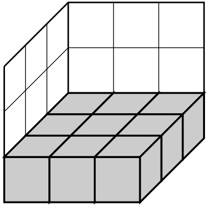
Slab:



Wall:



Layer:



Essentially, given the coordinates of all rook positions, no pair of rooks can have the same position in the same component. (Ex. rook positions (1, 2, 3) and (2, 3, 1) are valid, but (1, 2, 3) and (3, 2, 1) are not as they have the same second component).

Theorem

Let $r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_k(B)x^k$ be the rook polynomial for the darkened squares of an $n \times n \times n$ board B. Then, the number of ways to place n rooks on the light squares of B is counted by

$$(n!)^2 - r_1(B)((n-1)!)^2 + r_2(B)((n-2)!)^2 + \cdots + (-1)^k r_k(B)((n-k)!)^2.$$

Theorem

The rook polynomial of an $m \times n \times r$ board (denoted $B_{m,n,r}$) where all positions are restricted is

$$R(x, B_{m,n,r}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) P(r,k) x^k$$

where $s = \min\{m, n, r\}$.

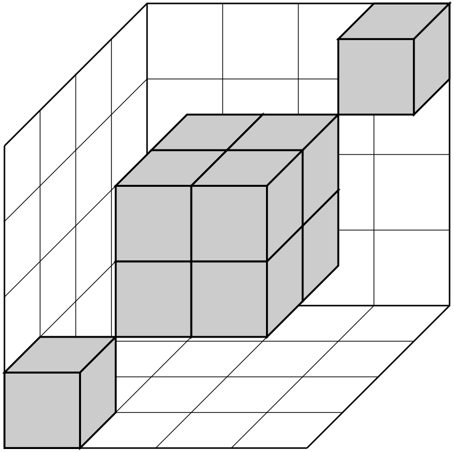
Proof

In a $m \times n \times r$ board there are $\binom{m}{k}$ ways to choose the slab, $P(n, k)$ ways to choose the wall, and $P(r, k)$ ways to choose the layer to place the k rooks.

We need to pick the wall and layer instead of choosing since the order we select them in matters.

For example, points (1, 2, 4) and (1, 3, 5) are different from points (1, 2, 5) and (1, 3, 4).

Example



Rook Polynomial of Subboards

We have disjoint subboards $B_{1,1,1}$, $B_{2,2,2}$, and $B_{1,1,1}$.

We can clearly see that $B_{1,1,1}$ has rook polynomial $R(x, B_{1,1,1}) = 1 + x$.

Using the formula for an $m \times n \times r$ board of all restricted squares, we get that the rook polynomial of $B_{2,2,2}$ is

$$\binom{2}{0} P(2,0)P(2,0) + \binom{2}{1} P(2,1)P(2,1)x + \binom{2}{2} P(2,2)P(2,2)x^2$$

giving $R(x, B_{2,2,2}) = 1 + 8x + 4x^2$.

Rook Polynomial

Then the rook polynomial of the original board B is

$$(1 + x)^2(1 + 8x + 4x^2) = 1 + 10x + 21x^2 + 16x^3 + x^4$$

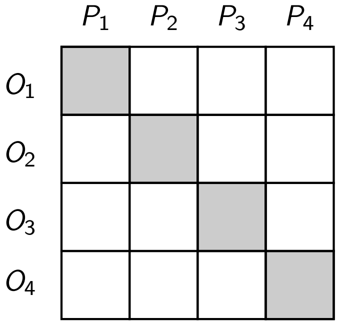
and we can place 4 non-capturing rooks on the light squares of the board in

$$(4!)^2 - 10 \times (3!)^2 + 21 \times (2!)^2 - 16 \times (1!)^2 + 1 \times (0!)^2 = 285$$

different valid arrangements.

Derangements in 2 and Higher Dimensions

A derangement of n objects in two dimensions is represented as a $n \times n$ board with the diagonal darkened.



Note that the squares along the diagonal are all disjoint, and have rook polynomial $1 + x$. Therefore any board representing a derangement of n objects has rook polynomial $(1 + x)^n$.

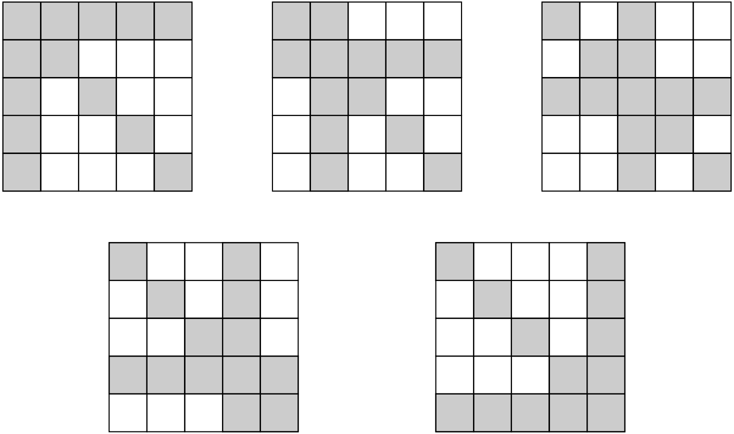
Generalization to Higher Dimensions

To extend the hat example, we consider k articles of clothing left by the door by each person.

Now, we are interested in arrangements where each person does not choose a single article of clothing that belongs to them, and for each item they choose, none of their other chosen items has that same owner.

Ex. if person A chooses person B's hat, person A may not choose person B's scarf as well. If we consider this in more dimensions than three, no pair of the m items a person chooses may belong to the same person.

Thee Dimensional Visualization



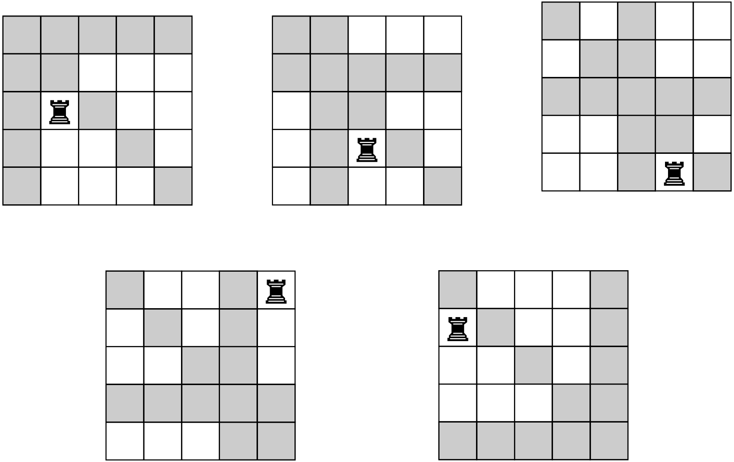
Connection to Latin Squares

There is a one-to-one correspondence between this generalized derangement of m people with $d-1$ articles of clothing, and Latin rectangles of size $d \times m$ with the first row in order.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

The above Latin rectangle gives rook coordinates (1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 1), (5, 1, 2).

Thee Dimensional Visualization



The End!

Benjamin Zindle. Rook polynomials for chessboards of two and three dimensions. 2007.

Feryal Alayont and Nicholas Krzywonos. Rook polynomials in three and higher dimensions. *Involve, a Journal of Mathematics*, 6(1):35–52, 2013.