Restricted Positions and Rook Polynomials

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1 Introduction

We wish to find the number of arrangements of n objects, where each object has a (possibly empty) set of restricted positions. We represent the restricted positions in an $n \times n$ board, where the rows represent objects and columns represent positions, and a darkened square represents a restricted position for an object. Then, finding an arrangement is equivalent to placing n markers on the white squares of our board in a way where no two markers are in the same row or column. Another way to describe this is by requiring the markers to be non-capturing rooks. In chess, rooks may move any number of squares either vertically or horizontally, thus a rook is non-capturing as long as no other pieces are in the same column or row.

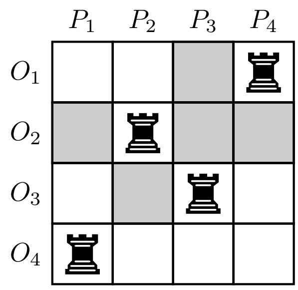


Figure 1: This represents the board for four objects O_1, O_2, O_3, O_4 where O_1 has restricted position P_3; O_2 has restricted positions P_1, P_2, P_3; O_3 has restricted position P_2; and O_4 has no restricted positions. Rooks placed in the manner shown represent the arrangement $O_4O_2O_3O_1$.

Definition (Rook Polynomial): Let $r_k(B)$ be the number of ways to place k non-capturing rooks on the *darkened squares* of a board B. The **rook polynomial** $R(x, B)$ is the generating function of $r_k(B)$:

$$R(x, B) = \sum_k r_k(B)x^k.$$

Note that for any any board B, $r_0(B) = 1$.

Note that while earlier we describe seeking a placement of rooks on the light squares, in general the rook polynomial is representing a placement of rooks on the darkened squares. Often, sources do not make a distinction between rook polynomial for the darkened squares vs. light squares. We will make this distinction by referring to the rook polynomial for placing rooks on darkened squares as the ‘rook polynomial for darkened squares’, and vice versa for placements of the rooks on light squares. If not specified, it is safe to make the assumption it refers to the placement of rooks on darkened squares.

Theorem: The rook polynomial for light squares of an $m \times n$ board (denoted $B_{m,n}$) with no restrictions is

$$R(x, B_{m,n}) = \sum_{k=0}^{s} \binom{m}{k} P(n, k)x^k$$

where $s = \min\{m, n\}$. Equivalently, this is also the rook polynomial of an $m \times n$ board where all squares are restricted.

Proof: Each term $r_k(x, B_{m,n}) = \binom{m}{k}P(n,k)$ counts the number of ways to place k non-capturing rooks on an $m \times n$ board with no restrictions. Each rook is in a unique column and row. There are $\binom{m}{k}$ ways to choose the rows, then $P(n,k)$ ways to choose and permute k of the columns. The columns are permuted instead of chosen as this represents the number of ways they can be paired with different rows. (Ex. placing rooks on squares (1, 2) and (2, 3) is a different arrangement than placing rooks on squares (1, 3) and (2, 2)).

The next theorem allows us to count the number of ways to place n rooks in the light squares of an $n \times n$ board with restrictions.

Theorem: Let $r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_k(B)x^k$ be the rook polynomial for the darkened squares of an $n \times n$ board B. Then, the number of ways to place n rooks on the light squares of B is counted by

$$n! - r_1(B)(n-1)! + r_2(B)(n-2)! + \cdots + (-1)^k r_k(B)(n-k)!. \tag{1}$$

Proof: We arrive at this formula through the Principle of Inclusion-Exclusion. Let P_i be the property that there are at least i rooks in forbidden positions. There are $r_i(B)$ ways to place i rooks in restricted positions. Then, there are $(n-i)!$ to arrange the rest of the rooks without consideration for whether their positions are restricted or not. Therefore for all $i \in \mathbb{N}$, $P_i = r_i(B')(n-i)!$, allowing us to arrive at the theorem above.

The next thing we can note is that permuting the rows and columns of a board B does not change the rook polynomial. This is just a matter of relabeling objects and positions, resulting in an equivalent problem. Therefore, we may permute the rows and columns to obtain disjoint subboards, where our board can be broken down into regions whose columns and rows to not overlap.

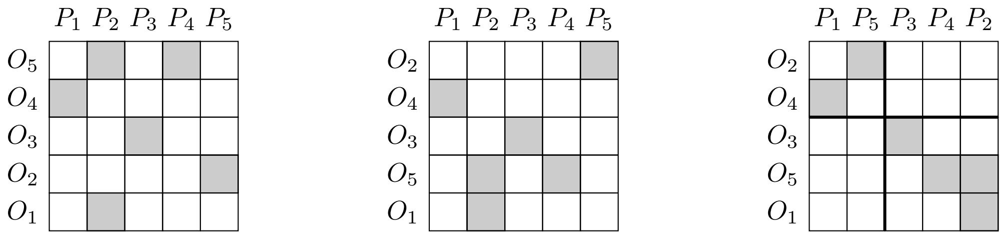


Figure 2: A demonstration of rearranging a board into two disjoint subboards. First rows O_2 and O_5 are exchanged, then columns P_2 and P_5 are exchanged to get two disjoint regions of darkened squares, one on columns P_1, P_5 and rows O_2, O_4, the other on columns P_3, P_4, P_2 and rows O_3, O_5, O_1.

Theorem (Disjoint Subboards): Let $R(x, B_1)$, $R(x, B_2)$ be the rook polynomials of disjoint boards B_1, B_2 of a board B. Then

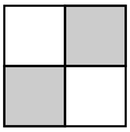
$$R(x, B) = R(x, B_1)R(x, B_2).$$

This idea can be extended to any number of disjoint subboards, thus

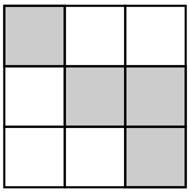
$$R(x, B) = R(x, B_1)R(x, B_2) \cdots R(x, B_k)$$

for disjoint subboards $B_1, \ldots, B_k$.

We may now easily find the rook polynomial for the darkened squares of the board in Figure 2.



The rook polynomial for the darkened squares of this board is $1 + 2x + x^2$ as there is one way to place zero rooks, two ways to place one rook, and one way to place two rooks on the darkened squares.



The rook polynomial for the darkened squares of this board is $1+4x+4x^2+x^3$ as there is one way to place zero rooks, four ways to place one rook, four ways to place two rooks, and one way to place three rooks on the darkened squares.

From this, we get that the rook polynomial $R(x, B)$ for the darkened squares of the board is

$$\begin{aligned} R(x, B) &= R(x, B_1)R(x, B_2) = (1 + 2x + x^2)(1 + 4x + 4x^2 + x^3) \\ &= 1 + 6x + 13x^2 + 13x^3 + 6x^4 + x^5. \end{aligned}$$

Now, we may use (1) to find the number of arrangements as

$$5! - 6 \times 4! + 13 \times 3! - 13 \times 2! + 6 \times 1! - 1 \times 0! = 33.$$

Therefore there are 33 ways to place rooks on the white squares of the original board in Figure 2 so that they are non-capturing.

1.1 Derangements

In a derangement, we seek an arrangement of objects such that each object is not in its original position. For example, in the hat problem, where we wish to know in how many ways a group of people who arrived with a hat can leave with any hat but their own. Given objects $O_1, \ldots, O_n$ ordered by their index, we want arrangements where the index of an objects position is not the same as the objects index, i.e. object O_i can be in any position but P_i. We can represent this as an $n \times n$ board with squares along the diagonal darkened.

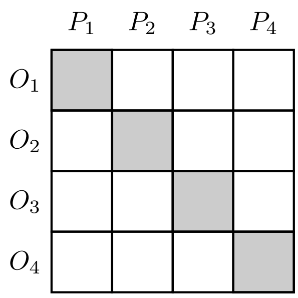


Figure 3: This board represents a derangement of four objects, where n non-capturing rooks are placed on the light squares.

Notice that each darkened square represents a disjoint subboard of size 1×1. The rook polynomial of each square is $1 + x$, making the rook polynomial of an $n \times n$ board $(1 + x)^n$.

Through the binomial theorem, we get that the rook polynomial of a derangement board is

$$R(x, B) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Then through expression (1) from the first section, which utilized the Principle of Inclusion-Exclusion, we are able to arrive at a formula for the number of derangements d_n:

$$d_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^n \binom{n}{n} 0!.$$

1.2 Triangle Boards and Stirling Numbers

A triangle board is a board where every square above the diagonal is restricted. This is equivalent to a permutation of n objects where O_1 cannot be in positions $P_2 \ldots P_n$; O_2 cannot be in positions $P_3 \ldots P_n$; O_3 cannot be in positions $P_4 \ldots P_n$; and repeat until object O_n, which has no restrictions on its position.

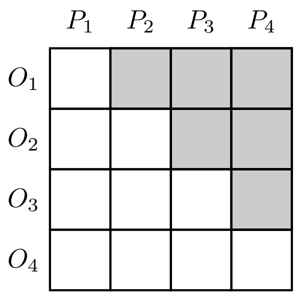


Figure 4: A triangle board of size 4.

A connection is made between triangle boards and the Stirling numbers of the second kind. The Stirling numbers of the second kind, $S(n,k)$, count the number of ways to partition n elements into k non-empty subsets. They are defined recursively as

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Theorem: The number of ways to place k non-capturing rooks on a triangle board of size m is equal to $S(m+1, m+1-k)$.

Proof: This proof closely follows that from [1], which uses induction on m.

Induction Hypothesis: Assume that $S(m+1, m+1-k)$ is the number of ways to place k non-capturing rooks on a triangle board of size m.

Basis of Induction: Start on a triangle board of size 2. If $k = 2$, then the only way to place rooks is along the diagonal (the highest non-restricted square of each column). There is 1 way to do this, and $S(2+1, 2+1-2) = S(3,1) = 1$. If $k = 1$, then there are three ways to place a single rook on a square as there are 3 squares. We can see that $S(2+1, 2+1-1) = S(3,2) = 3$. Therefore the basis of induction holds.

Induction Step: Consider placing k rooks ($0 \leq k \leq m+1$) on a triangle board of size $m+1$.

First of all, if $k = m+1$, then this can only be done through placing the rooks along the diagonal (top of each column), and if $k = 0$ then there is only one way as well. This provides the correct number as $S(m+2, m+2-1-(m+1)) = S(m+2, 1) = 1$ and $S(m+2, m+2-1-0) = S(m+2, m+1) = 1$.

Now we may consider the remaining possibilities, where $1 \leq k \leq m$. There are two cases:

Case 1: There are no rooks in the row with $m+1$ squares. If we remove this row we are left with a triangle board of size m, and we know there are $S(m+1, m+1-k)$ ways to place the

k rooks on this board from the induction hypothesis. Keep in mind that $k \leq m$, allowing us to place these k rooks on this smaller board.

Case 2: There is a rook in the row with $m+1$ squares. From the induction hypothesis there are $S(m+1, m+1-(k-1)) = S(m+1, m-k+2)$ ways to place the $k-1$ rooks on all rows except the largest one. We know there is exactly one rook in the largest row as there cannot be more than one in the same row. Since each of the $k-1$ rooks are in different columns (cannot be more than one in a column), there are $k-1$ squares in the bottom row in which we are unable to place our rook. Since there are $m+1$ squares in the largest row, we have $m+1-(k-1) = m-k+2$ available squares, giving a total of $(m-k+2)S(m+1, m-k+2)$ valid rook arrangements.

Summing all of these options results in

$$S(m+1, m+1-k) + (m-k+2)S(m+1, m+2-k)$$

arrangements. Comparing this to the recursive definition of the Stirling numbers of the second kind, we can observe that

$$\begin{aligned} S(m+2, m+2-k) &= S((m+2)-1, (m+2-k)-1) + (m+2-k)S((m+2)-1, m+2-k) \\ &= S(m+1, m+1-k) + (m-k+2)S(m+1, m+2-k) \end{aligned}$$

proving that the induction hypothesis holds for triangle boards of size $m+1$ and all $0 \leq k \leq m+1$.

It follows that $S(m+1, m+1-k)$ is the number of ways to place k non-capturing rooks on a triangle board of size m.

2 Extending to Three Dimensions

To extend this concept to three dimensions, we imagine a series of boards stacked upon each other, and will denote a series of squares along the new dimension as a 'tower', as described and originally introduced in [3].

We use the notion of 'slab', 'wall' and 'layer' from [1]. A slab refers to a set of squares with the same first component (rows in two dimensions), a wall refers to a set of squares with the same second component (columns in two dimensions), and a layer is a set of squares with the same third component. The distinction between slabs, walls, and layers, vs rows columns, and towers is that the former set is two-dimensional while the later refers to a single 'line' of blocks.

We now need to extend the movement of the rook. Now, the rook may move to any point which shares one (or more) components with it, meaning it may technically move diagonally so long as one component remains the same. This does make the rook analogy fall apart as it violates the rules of 3D chess [2], but for the sake of the interesting applications we are meant to ignore this inconsistency.

In the following theorems we will see that many ideas from the previous section extend quite easily to boards of three dimensions.

Theorem: The rook polynomial for light squares of an $m \times n \times r$ board (denoted $B_{m,n,r}$) with no restrictions (or equivalently, the rook polynomial for darkened squares for a board where all positions are restricted) is

$$R(x, B_{m,n,r}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) P(r,k) x^k \tag{2}$$

where $s = \min\{m, n, r\}$.

Proof: This extends the related theorem for two dimensions. Where in two dimensions there was $\binom{m}{k}$ ways to choose k rows, then $P(n,k)$ ways to choose and permute k columns, we next consider the number of ways to choose and permute k towers, which there is $P(r,k)$ ways to do, where again the permutation is done to assign the chosen towers to different row and column combinations. (Once again, providing the distinction between points (1,2,3), (2,3,4) and points (1,2,4), (2,3,3)).

Theorem: Let $r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_k(B)x^k$ be the rook polynomial for the darkened squares of an $n \times n \times n$ board B. Then, the number of ways to place n rooks on the light squares of B is counted by

$$(n!)^2 - r_1(B)((n-1)!)^2 + r_2(B)((n-2)!)^2 + \cdots + (-1)^k r_k(B)((n-k)!)^2. \tag{3}$$

Proof: The only distinction from the two-dimensional version of this proof is the squaring of the factorials. After choosing k restricted squares through $r_k(B)$, we now have k slabs and k walls we cannot place the remaining $n-k$ rooks in, providing us $n-k$ of each left to choose from. There are $(n-k)!$ ways to choose slabs, $(n-k)!$ ways to choose walls, resulting in the term being $r_k(B)((n-k)!)^2$.

Theorem (Disjoint Subboards): Subboards $B_1, B_2, \ldots, B_k$ of a three dimensional board B are disjoint if they share no slabs, walls, or layers. Then

$$R(x,B) = R(x,B_1)R(x,B_2)\cdots R(x,B_k).$$

We may use the previous theorems to find the number of ways to place non-capturing 'rooks' in the light blocks of the following board.

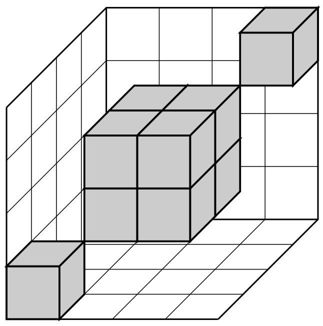


Figure 5: A three-dimensional board.

Notice that this board can be broken into three disjoint subboards. The board breaks down into two $1 \times 1 \times 1$ subboards, and the $2 \times 2 \times 2$ subboard in the center. We may use the formula in (2) to find the rook polynomials for these 'cube' subboards. We can easily see that the rook polynomials for the $1 \times 1 \times 1$ cubes are $1 + x$. For the $2 \times 2 \times 2$ cube we get:

$$\begin{aligned} R(x, B_{2,2,2}) &= \binom{2}{0} P(2,0)P(2,0) + \binom{2}{1} P(2,1)P(2,1)x + \binom{2}{2} P(2,2)P(2,2)x^2 \\ &= 1 + 8x + 4x^2. \end{aligned}$$

Then, we find the rook polynomial of B as

$$R(x,B) = (1+x)^2(1+8x+4x^2) = 1 + 10x + 21x^2 + 16x^3 + x^4.$$

Next we use (3) to arrive at the number of ways to place 4 rooks on the board:

$$(4!)^2 - 10 \times (3!)^2 + 21 \times (2!)^2 - 16 \times (1!)^2 + 1 \times (0!)^2 = 285.$$

Therefore there are 285 ways to place 4 rooks on the board from Figure 5.

2.1 Derangements in 3 (or more) Dimensions and Latin Squares

Now, we will generalize derangements to three or more dimensions. To extend upon the hat example, say there are d articles of clothing each person has left by the door (perhaps it is winter in Canada). Now, we are interested in arrangements where each person does not choose a single article of clothing that belongs to them, and for each item they choose, none of their other chosen items has that same owner. Ex. if person A chooses person B's hat, person A may not choose person B's scarf as well. If we consider this in more dimensions than three, no pair of the m items a person chooses may belong to the same person.

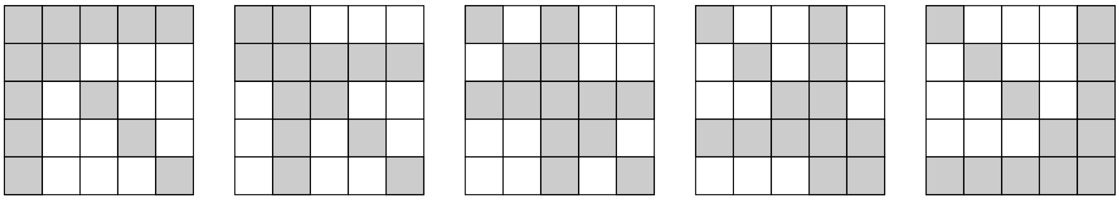


Figure 6: The cross-sections of a 3-dimensional derangement, equivalent to the clothing problem with 5 people and 3 articles of clothing.

Theorem: There is a one-to-one correspondence between this generalized derangement of m objects in d dimensions, and Latin rectangles of size $d \times m$ with the first row in order.

Proof: We will demonstrate that a bijection exists between these two structures.

First, consider the mapping from rook placements to Latin rectangles. Every rook placement has a unique coordinate, $(i_1, i_2, \ldots, i_d)$, representing its position in all d dimensions, where each of $i_1, i_2, \ldots, i_d \in \{1, \ldots, m\}$. Let $1, \ldots, m$ be the symbols of the Latin rectangle, so that we have a natural correspondence between coordinates and symbols in the rectangle. Since any two rooks are 'capturing' if they have any components in common, we know that no component is the same between any pair of coordinates. For example, if the symbol 1 appears as the first component (i_1) of one rook's coordinates, no other rook can have a 1 as the first component of their coordinates. Therefore, if we look at the same component in all coordinates, they will all be unique. Since we have m coordinates, we have m distinct symbols across rook coordinates for each component.

Map the coordinates to a position in the first row of the Latin rectangle based on their first component, in order from $1, \ldots, m$. For the second row, place the second component for each coordinate below their first component, and repeat for all components, filling in a $d \times m$ Latin rectangle. Since each component goes along each row of the Latin square, we have each symbol exactly once in each row. Since each coordinate will have no two components that are the same, the columns of the Latin rectangle will contain unique symbols in all squares. Thus this is a valid Latin rectangle and provides an injective mapping from rook placements to Latin rectangles.

Next, take any $d \times m$ Latin rectangle. Take the columns as the coordinates of rooks, where the first row is the first component, second row is the second component, etc. Since each column has no repeated symbols, each coordinate has no components that are the same. This is equivalent to no person choosing an item that belongs to them, as one of the components represents their 'restricted index' (this is the added third component in the 3-dimensional case), and also restricts them from choosing more than one item belonging to the same person. This covers all restrictions placed on choosing items.

Then, since rows have distinct elements, this means that in all coordinates there are none that have the same symbol in the same component. This is necessary in the example, where each component can be thought of representing a specific item or a person (their 'restricted index'). If one component has the same symbol between coordinates, this represents either two people choosing the same item, or two people having the same restricted index (i.e. they own the same items). As these are against the nature of the problem, this restriction is necessary. This covers all necessary restrictions in the problem, and demonstrated that Latin rectangles correspond quite naturally to this generalized problem of derangements.

Since we may map all rook placements in this generalized derangement problem and Latin rectangles to each other, this demonstrates a bijection between the two structures.

References

[1] Feryal Alayont and Nicholas Krzywonos. Rook polynomials in three and higher dimensions. *Involve, a Journal of Mathematics*, 6(1):35–52, 2013.

[2] Wikipedia. Three-dimensional chess — Wikipedia, the free encyclopedia, 2020. [Online; accessed 14-December-2020].

[3] Benjamin Zindle. Rook polynomials for chessboards of two and three dimensions. 2007.